TR/16/85 May 1985

The Extreme Residuals in Logistic Regression Models

by

Z. Al-Sarraf and D. H. Young

Summary

Goodness of fit tests for logistic regression models using extreme residuals are considered. Moment properties of the Pearson residuals are developed and used to define modified residuals, for the cases when the model fit is made by maximum likelihood, minimum chi-square and weighted least squares. Approximations to the critical values of the extreme statistics based on the ordinary and modified Pearson residuals are developed and assessed for the case when the logistic regression model has a single explanatory variable.

Contents

I. Introduction

In the statistical analysis of binary response data when explanatory variables are present, models such as the logistic, probit and complementary log-log are commonly used. The logistic model is particularly popular because of its simple interpretation in terms of odds ratios and here we shall restrict attention to this model.

We suppose that there are k explanatory variables having g distinct groups of values, which for the ith group are denoted by $x_{i1}, \ldots, x_{ik}$. For the ith group we suppose that n_i independent trials are made and let Y_i denote the number of "successes", i = 1,...g. If P_i denotes the true probability of success for each trial in the ith group, the linear logistic regression model is

$$\log(P_i/Q_i) = \underset{\sim}{x}_i' \underset{\sim}{\beta}, \qquad i = 1, \ldots, g \tag{1.1}$$

where $Q_i = 1 - P_i$, $\underset{\sim}{x}_i' = (1, x_{i1}, \ldots, x_{ik})$ and $\underset{\sim}{\beta}' = (\beta_0, \beta_1, \ldots, \beta_k)$ is a vector of unknown regression coefficients.

Three methods are commonly used for the estimation of $\underset{\sim}{\beta}$, namely maximum likelihood (ML), minimum chi-square (MCS) and weighted least squares (WLS), the last method being sometimes referred to as minimum logit chi-square. For later work it is helpful to consider these methods as members of a class of procedures in which the estimator $\hat{\underset{\sim}{\beta}}$ is found by minimising a function of the form

$$\phi = \sum_{i=1}^{g} n_i \phi_i (p_i, P_i) \tag{1.2}$$

where $p_i = Y_i/n_i$ is the sample proportion of successes for the ith group and $f_i(p_i, P_i)$ serves to measure the 'distance' between p_i and P_i. The forms of $f_1, (p_i, P_i)$ for the ML, MCS and WLS estimation procedures are

$$\phi_i^{(1)} = -(p_i \log P_i + q_i \log Q_i) \tag{1.3}$$

$$\phi_i^{(2)} = (P_i Q_i) - (p_i - P_i)^2 \tag{1.4}$$

$$\phi_i^{(3)} = p_i q_i \{\log(p_i/q_i) - \log(P_i/Q_i)\}^2 \tag{1.5}$$

respectively, where $q_i = 1 - p_i$.

The ML and MCS methods both require an iterative solution to estimate $\underset{\sim}{\beta}$ but the WLS procedure yields a non-iterative procedure. Further, in the case of a single explanatory variable there is now considerable evidence that the WLS procedure leads to estimates with better variance and mean square error properties (Berkson (1955), Al—Sarraf and Young (1985))- The WLS method can be applied when $p_i = 0$ or 1 but because the sample logit $z_i = \log(p_i/q_i)$ is undefined for these extreme cases, modified logits defined by

$$z_1^* = \log \{p_i + \frac{1}{2\,ni})/(q_i + \frac{1}{2\,ni})\}, \qquad i = 1,\ldots g \tag{1.6}$$

are sometimes used. We $E(z_i^*) = \underset{\sim}{x}_i'\underset{\sim}{\beta} + 0(N_i^{-2}$ and an estimate of the asymptotic variance of z_i^* which has very small bias is from Gart and Zweifel (1967)

$$\frac{(ni-1)(n_i+2}{n_i^3(pi+n_i^{-1})(q_i+n_i^{-1})} = w_1^{*-1} \quad \text{say .} \tag{1.7}$$

A modified weighted least squares (MWLS) estimate is then given by the value of $\underset{\sim}{\beta}$ which minimises $(\underset{\sim}{z}^* - \underset{\sim}{X}\beta)'\underset{\sim}{W}^*(\underset{\sim}{z}^* - \underset{\sim}{X}\beta$ where $\underset{\sim}{z}^{*\prime} = (z_1^*,\ldots,z_g^*)$ and $\underset{\sim}{W}^* = \text{diag}((w_1^*,\ldots,w_g^*))$.

In applications, it is of course important to assess the goodness of fit of the logistic regression model. This is commonly done by computing an overall summary statistic such as the Pearson statistic

$$R = \sum_{i=1}^{g} n_i (p_i - \hat{P}_i)^2/\hat{P}_i\hat{Q}_i) \tag{1.8}$$

where we use $P_i = \exp(\underset{\sim}{x}_i'\hat{\underset{\sim}{\beta}})/\{1+\exp(\underset{\sim}{x}'\hat{\underset{\sim}{\beta}})\}$ to denote the estimator of P_i under a general estimation procedure within the class defined in (1.2). An alternative summary statistic is the deviance statistic defined by

$$D = 2\sum_{i=1}^{g} n_i \{p_i \log(p_i/\hat{P}_i) + q_i \log(q_i/\hat{Q}_i)\} \quad , \tag{1.9}$$

although this statistic is only likely to be used if a maximum likelihood fit has been made. The individual group residuals corresponding to these overall statistics are given by

$$R_i = (n_i/\hat{P}_i\hat{Q}_i)^{\frac{1}{2}}(p_i - \hat{P}_i), \quad D_i = +2^{\frac{1}{2}} n_i^{\frac{1}{2}} \{p_i \log(p_i/\hat{P}_i) + q_i \log(q_i/\hat{Q}_i)\}^{\frac{1}{2}} \tag{1.10}$$

where $(\pm)$ is the sign of $p_i - P_i$, and an assessment of goodness of fit is often based on an inspection of these residuals or of a normal probability plot based on them.

The extreme residuals denoted by

$$R_{max} = \max_i R_i \,,\ R_{min} = \min_i R_i, \quad R_m = \max_i |R_i| \tag{1.11}$$

$$D_{max} = \max_i D_i \,,\ D_{min} = \min_i D_i \,,\ D_m = \max_i |D_i| \tag{1.12}$$

are themselves of particular interest and we focus attention on them in this study. A simple and common approach in assessing the extreme residuals appears to be to take them as being approximately distributed as the corresponding extremes in a sample of g independent observations from the N(0,1) distribution. This approach uses only the first order approximations to the mean and variance of the residuals and ignores their correlations, and so can be misleading.

In section 2, second order approximations to the expectations and covariance matrix of the Pearson residuals are given and used to define modified extreme residual statistics. Approximations to the percentage points of the extreme residual statistics are presented in section 3. Finally, results from a Monte Carlo investigation to assess the adequacy of the approximations are given for the case when there is a single explanatory variable and for various success probability configurations.

2. Approximations To The Moments Of The Pearson Residuals

In this section, we first derive approximations correct to $0(N^{-\frac{1}{2}})$ for the expectations of the Pearson residuals, where $N\sum_{i}^{g} n_i$ is the total number of trials. Our approach is very similar to the general approach given by Cox and Snell (1968), but here we assume that g is fixed and the $\{n_i\}$ are large, whereas in their method neglected terms are $0(g^{-1})$.

Since P. = $P_i(\text{ß})$, we may write

$$R_i = h_i(p_i, \hat{\underset{\sim}{\beta}}), \qquad \varepsilon_i = h_i(p_i, \underset{\sim}{\beta}) \tag{2.1}$$

Where

$$h_i(p_i, \underset{\sim}{\beta} = \left(\frac{n_i}{P_i Q_i}\right)^{\frac{1}{2}} \left\{ P_i - \frac{e^{\underset{\sim}{x}'_i \underset{\sim}{\beta}}}{1 + e^{\underset{\sim}{x}'_i \underset{\sim}{\beta}}} \right\} \tag{2.2}$$

A Taylor series expansion about 3 gives to second order

$$R_i = \varepsilon_i + \{(P_i - \frac{1}{2})\varepsilon_i - (n_i P_i Q_i)^{\frac{1}{2}}\} \sum_r x_{ir} (\hat{\beta}_r - \beta_r) \tag{2.3}$$

so

$$E(R_i) \approx (n_i P_i Q_i)^{\frac{1}{2}}\} \sum_r x_{ir} \{(P_i - \frac{1}{2})(P_i Q_i)^{-1} a_{ir} - b_r\} \tag{2.4}$$

where b_r denotes the bias of ß_r and

$$a_{ir} = E\{p_i - P_i)(\hat{\beta}_r - \beta_r)\} \tag{2.5}$$

The biases have been found by Sarraf and Young (1985) correct to $0(N^{-1})$ for the ML, MCS and WLS estimation procedures and are given by

$$b_r^{(1)} = -\frac{1}{2} \sum_s \sum_t \sum_u I^{rs} I^{tu} K_{stu} \tag{2.6}$$

$$b_r^{(2)} = b_r^{(3)} = \frac{1}{2} \sum_s I^{rs} \sum_i x_{is} (Q_i - P_i) + 2b_r^{(1)} \tag{2.7}$$

respectively, where

$$I_{rs} = \sum_i n_i P_i Q_i x_{ir} x_{is}, \qquad K_{rst} = \sum_i n_i P_i Q_i (Q_i - P_i) x_{ir} x_{is} x_{it} \tag{2.8}$$

and I^{rs} denotes the element in the (r+1) st row and (s + 1) column in the inverse of $\underset{\sim}{I} = ((I_{rs}))$. All summations over r.s,... run from 0,1,....k.

To find a_{ir} correct to $0(N^{-1})$, we may use the first order approximation

$$\hat{\beta}_r - \beta_r = -\sum_s \lambda^{rs} U_s \tag{2.9}$$

where

$$U_r = \frac{\partial \phi}{\partial \beta_r}, \qquad \lambda_{rs} = E\left(\frac{\partial^2 \phi}{\partial \beta_r \, \partial \beta_s}\right) \tag{2.10}$$

and λ_{rs} denotes the element in the (r+1)st row and (s+1)st column of

$\underset{\sim}{\lambda} = ((\lambda_{rs}))$.

For ML estimation, we have

$$U_r = -\sum_i n_i x_{ir} (p_i - P_i), \quad \lambda_{rs} = I_{rs}, \quad E\{(p_i - P_i) U_s\} = - \text{xis}\, P_i Q_i$$

so correct to $0(N^{-1})$ we have

$$a_{ir}^{(1)} = \sum_s I^{rs} x_{is} P_i Q_i \tag{2.11}$$

For MCS estimation, we have

$$U_r = -2 \sum_i n_i x_{ir} (pi - P_i) - (P_i - \tfrac{1}{2})(p_i - P_i)^2 (P_i Q_i)^{-1} \quad ,$$

$$\lambda_{rs} = 2I_{rs} + 0(1), \qquad E\{(p_i - P_i) U_s\} = -2x_{is} P_i Q_i + 0(n_i^{-1}) \ .$$

Hence correct to $0(N^{-1})$ we have

$$a_{ir}^{(1)} = \sum_s I^{rs} x_{is} P_i Q_i \tag{2.12}$$

For WLS estimation, we have

$$U_r = -2 \sum_i n_i x_{ir} \, p_i - q_i) \ \{\log (P_i/q_i - \log (P_i / Q_i)\}$$

$$\lambda_{rs} = 2 \ I_{rs} + 0(1), \qquad E\{(p_i - P_i) U_s\} = -2x_{is} P_i Q_i + 0(n_i^{-1}) \ .$$

giving

$$a_{ir}^{(1)} = \sum_s I^{rs} x_{is} P_i Q_i \tag{2.13}$$

Since $a_{ir}^{(1)} = a_{ir}^{(2)} = a_{ir}^{(3)}$, we have the general formula correct to $0(N^{-\frac{1}{2}}$ covering the three methods of estimation

$$E(R_i) \approx (n_i P_i Q_i)^{\frac{1}{2}} \{(P_i - \tfrac{1}{2}) \sum_r \sum_s x_{ir} x_{is} I^{rs} - \sum_r x_{ir} b_r\}, \tag{2.14}$$

where the biases b are given by (2.6) and (2.7).

The second order expression for the covariance matrix of the Pearson residuals is given by

$$\text{cov}(\underset{\sim}{R}) \; \underset{\sim g}{I} - \underset{\sim}{n} \underset{\sim}{v}^{-\frac{1}{2}} \underset{\sim}{P}^{*'} \underset{\sim}{I}^{-1} \underset{\sim}{P}^{*} \underset{\sim}{V}^{-\frac{1}{2}} \underset{\sim}{n} \tag{2.15}$$

where $\underset{\sim}{R}' = (R_1 \ldots, R_g)$, $\underset{\sim}{n} = \text{diag}(n_1, \ldots, n_g)$, $\underset{\sim}{V} = \text{diag}(n_1 P_1 Q_1, \ldots, n_g P_g Q_g)$, $\underset{\sim g}{I}$ is the identity matrix of order g and

$$\underset{\sim}{P}^{*} = \begin{bmatrix} \partial P_1/\partial\beta_0 \ldots \partial P_g/\partial\beta_0 \\ \partial P_1/\partial\beta_k \ldots \partial P_g/\partial\beta_k \end{bmatrix} \tag{2.16}$$

This covariance result holds for ML, MCS and WLS estimation and for arbitrary model specification for the {P.}. For the logistic regression model we have $\partial P_i/\partial \beta_j = x_{ij} P_i Q_i$ so $\underset{\sim}{P}^{*} = \underset{\sim}{X}' \underset{\sim}{n}^{-1} \underset{\sim}{V}$ and hence

$$\text{cov}(\underset{\sim}{R}) \; \underset{\sim g}{I} - \underset{\sim}{V}^{-\frac{1}{2}} \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{V}\underset{\sim}{X})^{-1} \underset{\sim}{X}'\underset{\sim}{V}^{-\frac{1}{2}} \tag{2.17}$$

If $c_{j\,j}$. denotes the (i.i)th element in $\underset{\sim}{C} = \underset{\sim}{V}^{-\frac{1}{2}} \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{V}\underset{\sim}{X})^{-1} \underset{\sim}{X}'\underset{\sim}{V}^{-\frac{1}{2}}$ then

Hence $C_{ij} = (n_i P_i Q_i)^{\frac{1}{2}} (n_j P_j Q_j)^{\frac{1}{2}} \sum_r \sum_s x_{ir} x_{is} I^{rs}$. (2.18)

$$\text{var}(R_i) = 1 - C_{ii} = -n_i P_i Q_i \sum_r \sum_s x_{ir} x_{is} I^{rs}. \tag{2.19}$$

and

$$\text{corr}(R_i, R_j) = -c_{ij}/\{(1-c_{ii})(1-c_{jj})\}^{\frac{1}{2}} \tag{2.20}$$

Although the variances of the $\{R_j\}$ depend on the $\{P_j\}$ which are unknown, their average variance is independent of the $\{P_i\}$. Thus using (2.17), the sum of the variances is given by

$$\begin{aligned} \sum_i \text{var}(R_i) &= \text{tr}(I_g) - \text{tr}\{V^{\frac{1}{2}} X(X'VX)^{-1} X'V^{\frac{1}{2}}\} \\ &= \text{tr}(\underset{\sim g}{I}) - \text{tr}(\underset{\sim k+1}{I}) = g - k - 1 \end{aligned} \tag{2.21}$$

In order to obtain information about the magnitude of the approximate expectations and variances given by (2.14) and (2,19), their values have been computed for the case of a single explanatory variable with $x_i = i - 1$, $i = 1,\dots,g$ for $g = 5, 10$, $n = 25, 50, 100$ and six (β_0, β_1) configurations giving a wide range of values for the success probabilities $\{P.\}$. The configurations are shown in table 1.

TABLE 1

Parameter values (β_0, β_1) and success probabilities

			$\{P_i\}$				
	$g = 5$						
(i)	$\beta_0 = -2.0$,	$\beta_1 = 0.4$	0.119,	0.168,	0.232,	0.310,	0.401
(ii)	$\beta_0 = -1.0$	$\beta_1 = 0.5$	0.269,	0.378,	0.500,	0.623,	0.731
(iii)	$\beta_0 = 0.5$,	$\beta_1 = 0.5$	0.623,	0.731,	0.818,	0.881,	0.924
	$g = 10$						
(iv)	$\beta_0 = -2.0$,	$\beta_1 = 0.2$	0.119,	0.142,	0.168,	0.198,	0.231
			0.269,	0.310,	0.354,	0.401,	0.450
(v)	$\beta_0 = -0.4$,	$\beta_1 = 0.2$	0.401,	0.450,	0.500,	0.550,	0.591
			0.646,	0.690,	0.731,	0.769,	0.802
(vi)	$\beta_0 = 0.5$,	$\beta_1 = 0.2$	0.623,	0.668,	0.711,	0.750,	0.785
			0.818,	0.846,	0.870,	0.891,	0.908

Values of $E(R_i)$ based on the ML, MCS and WLS methods of estimation are shown in table 2 for the six configurations given in table 1 and for sample sizes $n. = n = 25, 50, 100$, $i=1,\dots,g$. Values of the approximate variances of the $\{R.\}$ given by (2.19) are also shown, these variances being independent of the common sample size n.

TABLE 2

Approximate expectations and variances of residuals.

$E_i^{(1)} = 10 \times E(R_i)$ based on ML fit, $E_i^{(2)}$ $10 \times E(R_i)$ based on MCS or WLS fit

		$n = 25$		$n = 50$		$n = 100$		
Configuration	i	$E_i^{(1)}$	$E_i^{(2)}$	$E_i^{(1)}$	$E_i^{(2)}$	$E_i^{(1)}$	$E_i^{(2)}$	$var(R_i)$
	1	-0.34	-1.62	-0.24	-1.15	-0.17	-0.81	0.504
	2	0.14	-1.00	0.10	-0.71	0.07	-0.50	0.671
(i)	3	0.28	-0.63	0.20	-0.45	0.14	-0.32	0.780
	4	0.09	-0.49	0.07	-0.35	0.05	-0.25	0.707
	5	-0.21	-0.39	-0.15	-0.28	-0.11	-0.20	0.338

Table 2 cont

	1	-0.07	-0.56	-0.05	-0.40	-0.07	-0.56	0.438
	2	0. 12	-0.15	0.08	-0.11	0. 12	-0. 15	0.674
(ii)	3	0 .00	0.00	0.00	0.00	0.00	0.00	0.775
	4	-0.12	0.15	-0.08	0.11	-0.12	0.15	0.674
	5	0.07	0.56	0.05	0.40	0.07	0.56	0.438
	1	0.28	0.39	0.20	0.27	0. 14	0. 19	0.309
	2	-0.16	0.63	-0.11	0.44	-0.08	0.31	0.710
(iii)	3	-0.3 7	0.92	-0.26	0.65	-0. 18	0.46	0.767
	4	-0.13	0. 15	-0.09	1.03	-0.06	0.73	0.663
	5	0.45	0.22	0.32	0.15	0.23	0.11	0.551
	1	-0.22	-1 .84	-0. 16	-1.30	-0.11	-0.92	0.733
	2	-0.08	-1 .63	-0.06	-1.15	-0.04	-0.81	0.774
	3	0.03	-1 .42	-0.02	-1 .01	0.02	-0.71	0.817
	4	0,10	-1 .23	0.07	-0.87	0.05	-0.61	0.855
(iv)	5	0.13	-1 .05	0.09	-0.74	0.07	-0.52	0.883
	6	0.12	-0.87	0.08	-0.62	0.06	-0.44	0.892
	7	0.07	-0.71	0.05	-0.50	0.03	-0.36	0.874
	8	0.00	-0.54	0.00	-0.38	0.00	-0.27	0.824
	9	-0.07	-0.35	-0.05	-0.25	-0.04	-0.18	0.737
	10	-0.11	-0. 13	-0.08	-0.09	-0.06	-0.06	0.612
	1	0.01	-0.32	-0.01	-0.22	-0.01	-0. 16	0.648
	2	0.05	-0.11	0.03	-0.08	0.02	-0.05	0.744
	3	0.04	0.07	0.03	0.05	0.02	0.03	0.819
	4	0.00	0.21	0.00	0. 15	0.00	0.1 1	0.869
(v)	5	-0.04	0.35	-0.03	0.25	-0.02	0.17	0.891
	6	-0.07	0.48	-0.05	0.34	-0.04	0.24	0.886
	7	-0.08	0.63	-0.06	0.45	-0.04	0.31	0.859
	8	-0.05	0.79	-0,03	0,56	-0.02	0.40	0.816
	9	-0.02	0.98	0.02	0.69	0.01	0.49	0.762
	10	0.13	0.12	0.09	0.83	0.07	0.59	0.706

Table 2 cont...

	1	0.17	0.34	0.12	0.24	0.09	0.17	0.596
	2	0.08	0.58	0.05	0.41	0.04	0.29	0.735
	3	-0.03	0.79	-0.02	0.56	-0.01	0.39	0.827
	4	-0.11	0.98	-0.08	0.69	-0.05	0.49	0.877
(vi)	5	-0.16	1.17	-0.11	0.83	-0.08	0.58	0.892
	6	-0.16	1 .36	-0.11	0.96	-0.08	0.68	0.881
	7	-0.11	1.57	-0.08	1.11	-0.06	0.79	0.854
	8	-0.02	1.79	-0.02	1.26	-0.01	0.89	0.818
	9	0.11	2.01	0.08	1.42	0.06	1.01	0.779
	10	0.27	2.23	0.19	1.58	0.14	1.12	0.743

The results show that the absolute values of the approximate expectations of the residuals based on a ML fit are generally much smaller than those based on a MCS or WLS fit. Also, the approximate variances of the residuals are often appreciably less than one, particularly for the extreme values of the index i of the residuals.

The approximations to the expectations and variances of the residuals allow modified residuals to be used whose mean and variance are likely to be closer to zero and one, respectively, than the corresponding moments of the unmodified residuals $\{R_i\}$.

Modified residuals allowing for variance adjustment are defined by

$$R_i^* = R_i/(1-\hat{C}_{ii})^{\frac{1}{2}}, \quad i = 1,---,g \tag{2.22}$$

Where

$$C_{ii} = -n_i\hat{P}_i\hat{Q}_i \sum_r \sum_s x_{ir}x_{is}\hat{I}^{rs} . \tag{2.23}$$

denotes the estimate of c_{ii} using the $\{P_i\}$.

With variance and expectation adjustment, the modified residuals are defined by

$$R_i^{**} = (R_i - \hat{E}_i) / (1-\hat{C}_{ii})^{\frac{1}{2}}, \quad i = 1,....,g \tag{2.24}$$

where $\hat{E}_i$ denotes the estimate of $E(R_i)$.

Using the modified residuals, extreme residual statistics will be denoted by

$$R^*_{max} = \max_i R_i^*, \quad R^*_{min} = \min_i R_i^*, \; R^*_m = \max_i |R_i^*| \tag{2.25}$$

$$R^{**}_{max} = \max_i R^{**}, \quad R^{**}_{min} = \min_i R_i^{**}, \; R^{**}_m = \max_i |R_i^{**}| \tag{2.26}$$

3. Approximations To The Critical Values of The Extreme Residual Statistics

In order to use the extreme residual statistics in formal goodness of fit tests for the logistic regression model, we need approximations to the percentage points of their distributions when the model is correct. Simple approximations are available based on the use of the Banferroni inequality. We illustrate the approach for the R_{max} statistic.

We have

$$\sum_i P(R_i \geq r) - \sum_i \sum_{<j} P(R_i \geq r, R_j \geq r) \leq P(R_{max} \geq r) \leq P(R_i \geq r) \tag{3.1}$$

Since nearly all pairs of residuals are negatively correlated, we are led to the conjecture that

$$\sum_i \sum_{<j} P(R \geq r, R_j \geq r) < \sum_i \sum_{<j} P(R_i \geq r, R_j \geq r) \tag{3.2}$$

This leads to the inequality

$$\sum_i P(R_i \geq r) - \frac{1}{2}\{\sum_i P(R_i \geq r)\}^2 \leq P(R_{max} \geq r) \leq \sum_i P(R_i \geq r) \tag{3.3}$$

For large r, we use the approximation

$$P(R_{max} \geq r) \approx \sum_i P(R_i \geq r) \tag{3.4}$$

the error in the approximation being less than $\frac{1}{2}(1-g^{-1})\{\sum_i P(R_i \geq r)\}^2$ if the conjective given by (3.2) is true. Since $var(R_i)^i$ given by (2.19) depends on the $\{P_i\}$ which are unknown, we may either use the estimates of $var(R_i)$ based on the fitted model or ignoring the variations in the variances work with their average value (g-k-1)/g. Using the latter approach we take

$$\left(\frac{g}{g-k-1}\right)^{\frac{1}{2}} R_i \underset{\sim}{\text{approx}} N(0,1) \tag{3.5}$$

and use of (3.4) gives

$$P(R_{max} \geq r) \approx 1 - \phi\left\{\left(\frac{g}{g-k-1}\right)^{\frac{1}{2}} r\right\} \tag{3.6}$$

where $\Phi(\cdot)$ denotes the c.d.f. of the N(0,1) distribution. If we let r_{max} (1-α) denote the upper 100a percentage point of the distribution of R_{max}, we have the approximation

$$r_{max}(1-\alpha) \approx \left(\frac{g-k-1}{g}\right)^{\frac{1}{2}} u_{1-\frac{\alpha}{g}} \tag{3.7}$$

where $u_{1-\alpha}$ is the 100(1-α) percentile of the N(0,1) distribution. If r_{min} (α) and r_m (100-α) denote the lower and upper 100α percentage points of the distributions of R_{min} and R_m respectively, similar arguments lead to the approximations

$$r_{max}\ (\alpha) \approx \left(\frac{g-k-1}{g}\right)^{\frac{1}{2}} u_{1-\frac{\alpha}{g}} \tag{3.8}$$

$$r_{max}\ (1-\alpha) \approx \left(\frac{g-k-1}{g}\right)^{\frac{1}{2}} u_{1-\frac{\alpha}{2g}} \tag{3.9}$$

Since to the same order of approximation var(D.) = var(R.), approximations to the critical values d_{max} (1-α), d_{min} (a) and d_m (1-α) for the D_{max}, D_{min} and D_m statistics are also given by (3.7), (3.8) and (3.9), respectively.

For the modified residuals R^*_i which use the variance estimates and R_i^{**} which use both expectation and variance estimates, we take both sets of residuals to be approximately distributed as N(0,1). If the logistic regression model is correct, we are led to the approximations

$$r^*_{max}\ (1-\alpha) = r^{**}_{max}\ (1-\alpha) \approx u_{1-\frac{\alpha}{g}} \tag{3.10}$$

$$r^*_{min}\ (1-\alpha) = r^{**}_{min}\ (1-\alpha) \approx u_{1-\frac{\alpha}{g}} \tag{3.11}$$

$$r^*_{m}\ (1-\alpha) = r^{**}_{m}\ (1-\alpha) \approx u_{1-\frac{\alpha}{2g}} \tag{3.12}$$

4. Monte Carlo Results

In order to examine the accuracy of the approximations to the critical values of the extreme residual statistics, a monte carlo investigation was made for the case of a single explanatory variable using the parameter configurations given in table 1. Equal sample sizes n. = n = 25, 50, 100, i = 1,...,g were used. The model fits by ML, MCS, WLS and MWLS estimation were made using the statistical package GLIM and a run-size of 2000 was used in each case. The empirical distributions of the extreme residual statistics were used to obtain the upper and lower critical values for significance levels α - 0.10, 0.05, 0.025 and 0.01. The actual significance levels associated with the approximating critical values given by (3.7) to (3.12) were also determined and contrasted with the nominal values of α .

The broad findings reached from the investigation are:

(i) the differences between the values of the extreme statistics based on the modified residuals R_i^* and R_i^{**} are in general very small so little is gained by making an additional adjustment for estimated expectation, once the ordinary residuals have been adjusted for estimated variance.

(ii) The differences between the significance levels when variance adjusted residuals are used instead of the ordinary Pearson residuals are generally quite small. For the extreme statistics based on the absolute values of the residuals, variance adjustment leads to some improvement for the smaller values of a .

(iii) No one estimation procedure systematically provides better control over the significance level of the test.

Tables 3 to 14 show the actual significance levels as estimated by simulation associated with the approximating values for the extreme residual statistics based on the ordinary residuals and the variance adjusted residuals. It is encouraging to see that there is generally good agreement with the nominal significance levels, particularly for the modulus statistics R_m and R_m^*.

Table 3

Estimated significance levels for approximate critical values for (a) R_{max} and (b) R^*_{max} , based on ML fit.

		$\alpha=0.10$		$\alpha=0.05$		$\alpha=0.025$		$\alpha=0.01$	
configuration		(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
	n=25	0.123	0.107	0.064	0.057	0.038	0.029	0.017	0.011
i)	n=50	0.116	0.101	0.070	0.055	0.039	0.029	0.018	0.013
	n=100	0.109	0.090	0.057	0.047	0.032	0.025	0.017	0.012
	n=25	0.109	0.101	0.056	0.050	0.031	0.022	0.015	0.009
ii)	n-50	0.089	0.088	0.052	0.042	0.026	0.022	0.012	0.011
	n=100	0.092	0.088	0.045	0.044	0.023	0.021	0.011	0.009
	n=25	0.067	0.071	0.031	0.030	0.014	0.010	0.004	0.004
iii)	n=50	0.084	0.081	0.043	0.040	0.019	0.016	0.007	0.005
	n-100	0.090	0.083	0.050	0.038	0.023	0.018	0.009	0.007
	n=25	0.142	0.137	0.082	0.080	0.046	0.044	0.021	0.019
iv)	n=50	0.114	0.112	0.060	0.055	0.026	0.023	0.012	0.011
	n=100	0.122	0.111	0.061	0.062	0.033	0.033	0.017	0.018
	n=25	0.083	0.079	0.049	0.042	0.018	0.016	0.007	0.007

Table 3 cont....

v)	n=50	0.088	0.092	0.042	0.040	0.018	0.018	0.007	0.009
	n=100	0.089	0.086	0.042	0.042	0.021	0.021	0.008	0.008
	n=25	0.044	0.043	0.020	0.019	0.008	0.008	0.001	0.002
vi)	n=50	0.070	0.070	0.029	0.030	0.012	0.010	0.002	0.002
	n=100	0.077	0.074	0.037	0.033	0.013	0.014	0.005	0.005

<u>Table 4</u>

Estimated significance levels for approximate critical values for (a) R_{min} and (b) $R^*_{min.}$, based on ML fit.

		α=0.10		α=0.05		α=0.025		α=0.01	
configuration		(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
	n-25	0,085	0.082	0.36	0.038	0.015	0.020	0.005	0.004
i)	n=50	0.087	0.079	0.042	0.038	0.019	0.018	0.008	0.004
	n=100	0.081	0.084	0.045	0.03 7	0.022	0.021	0.012	0.007
	n=25	0.097	0.095	0.053	0.046	0.026	0.024	0.012	0.009
ii)	n=50	0.097	0.091	0.050	0.049	0.028	0.025	0.013	0.010
	n=100	0.010	0.088	0.047	0.044	0.025	0.023	0.011	0.008
	n=25	0.141	0. 124	0.080	0.061	0.043	0.038	0.025	0.015
iii)	n=50	0.128	0. 1 10	0.074	0.06 1	0.04 1	0.029	0.019	0.012
	n=100	0.100	0.089	0.055	0.043	0.026	0.021	0.012	0.008
	n=25	0.067	0.065	0.029	0.029	0.012	0.012	0.005	0.003
iv)	n=50	0.078	0.075	0.028	0.026	0.013	0.011	0.005	0.003
	n=100	0.081	0.078	0.034	0.033	0.013	0.015	0.005	0.004
	n=25	0.100	0.095	0.047	0.04 7	0.026	0.025	0.016	0.014
v)	n=50	0.114	0.112	0.058	0.058	0.031	0.029	0.011	0.010
	n=100	0.109	0. 103	0.057	0.050	0.032	0.027	0.013	0.012
	n=25	0. 160	0.150	0.095	0.086	0.057	0.052	0.026	0.026
vi)	n=50	0.130	0.128	0.072	0.06 7	0.039	0.035	0.018	0.014
	n=100	0,132	0.125	0.078	0.073	0.046	0.041	0.024	0.019

Table 5

Estimated significance levels for approximate critical values for (a) R_m and (b) R_m statistics, based on ML fit.

Configuration		α=0.10 (a)	α=0.10 (b)	α=0.05 (a)	α=0.05 (b)	α=0.025 (a)	α=0.025 (b)	α=0.01 (a)	α=0.01 (b)
	n=25	0.09 1	0.077	0.050	0.041	0.029	0.018	0.012	0.006
i)	n=50	0-101	0.079	0.055	0.042	0.032	0.019	0.014	0.009
	n=100	0.089	0.069	0.049	0.037	0.032	0.019	0.016	0.010
	n=25	0.096	0.081	0.052	0.040	0.029	0.020	0.013	0.008
ii)	n=50	0.089	0.075	0.049	0.040	0.029	0.022	0.028	0.010
	n=100	0.079	0.076	0.043	0.039	0.024	0.020	0.014	0.007
	n=25	0.101	0.074	0.054	0.043	0.034	0.023	0.016	0.013
iii)	n=50	0. 108	0.086	0.058	0.039	0.030	0.017	0.012	0.009
	n=100	0.096	0.069	0.045	0.033	0.026	0.016	0.008	0.006
	n=25	0.106	0, 104	0.058	0.056	0.03 1	0.026	0.014	0.012
iv)	n=50	0.085	0.078	0.038	0.034	0.019	0.017	0.009	0.006
	n=100	0.093	0.092	0.045	0.047	0.026	0.025	0.013	0.011
	n=25	0.083	0.084	0.042	0.041	0.026	0.024	0.012	0.008
v)	n=50	0.094	0.090	0.047	0.043	0.022	0.022	0.007	0.007
	n=100	0.093	0.086	0.049	0.046	0.027	0.023	0.012	0.009
	n=25	0. 112	0. 101	0.064	0.060	0.034	0.030	0.018	0.018
vi)	n=50	0.096	0.094	0.049	0.044	0.025	0.022	0.010	0.007
	n=100	0.111	0.102	0.058	0.054	0.036	0.030	0.014	0.013

Table 6

Estimated significance levels for approximate critical values for (a) R_{max} and (b) R_{max}* statistics p, based on MCS fit.

	α-0.10		α=0.05		α=0.025		α=0.01	
Configuration	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
n=25	0.152	0.046	0.02 7	0.020	0.014	0.009	0.004	0.003
i) n=50	0.137	0.058	0.033	0.027	0.017	0.011	0.009	0.005
n=100	0.113	0.062	0.038	0.027	0.019	0.014	0.008	0.006
n=25	0.104	0.100	0.057	0.051	0.027	0.022	0.014	0.011
ii) n=50	0.094	0.087	0.052	0.049	0.025	0.022	0.011	0.010
n=100	0.096	0.088	0.046	0.044	0.023	0.019	0.012	0.010
n=25	0.039	0. 136	0.065	0.069	0.028	0.032	0.013	0.017
iii) n=50	0.056	0.132	0.071	0.074	0.040	0.053	0.014	0.01 1
n=100	0.053	0.1 13	0.073	0.066	0.041	0.031	0.017	0.013
n=25	0.125	0.080	0.036	0.038	0.017	0.017	0.009	0.009
iv) n=50	0.117	0.064	0.030	0.026	0.014	0.013	0.005	0.005
n=100	0.102	0.084	0.044	0.042	0.022	0.022	0.012	0.010
n=25	0.075	0.091	0.051	0.045	0.027	0.020	0.009	0.009
v) n=50	0.093	0.102	0.050	0.053	0.023	0.024	0.008	0.009
n=100	0.09 7	0.101	0.051	0.050	0.023	0.023	0.010	0.010
n=25	0.061	0.074	0.030	0.030	0.012	0.017	0.004	0.004
vi) n=50	0.071	0. 109	0.051	0.049	0.021	0.024	0.005	0.005
n=100	0.094	0.106	0.052	0.04 7	0.025	0.024	0.009	0.008

Table 7

Estimated significance levels for approximate critical values for (a) R_{min} and (b) R_{min}^* statistics, based on MCS fit.

Configuration		α=0.10 (a)	(b)	α=0.05 (a)	(b)	α=0.025 (a)	(b)	a=0.01 (a)	(b)
	n=25	0.124	0.129	0.071	0.069	0.034	0.040	0.012	0.019
i)	n=50	0.127	0.122	0.065	0.062	0.035	0.036	0.018	0.016
	n=100	0.109	0.109	0.063	0.061	0.034	0.035	0.017	0.015
	n=25	0.100	0.097	0.055	0.051	0.028	0.023	0.010	0.01 1
ii)	n=50	0.099	0.087	0.051	0.049	0.031	0.026	0.014	0.013
	n=100	0.095	0.090	0.047	0.045	0.025	0.021	0.013	0.009
	n=25	0.048	0.041	0.025	0.017	0.014	0.008	0.007	0.002
iii)	n=50	0.063	0.048	0.031	0.020	0.015	0.009	0.005	0.002
	n=100	0.056	0.046	0.027	0.018	0.011	0.007	0.005	0.003
	n=25	0. 1 00	0.098	0.047	0.045	0.020	0.020	0.007	0.007
iv)	n=50	0.108	0. 101	0.049	0.045	0.017	0.018	0.008	0.006
	n=100	0.086	0.099	0.045	0.042	0.022	0.021	0.006	0.005
	n=25	0.075	0.074	0.039	0.037	0.020	0.020	0.009	0.006
v)	n=50	0.094	0,095	0.049	0.049	0.023	0.020	0.008	0.004
	n=100	0. 100	0.091	0.046	0.044	0.025	0.023	0.012	0.010
	n=25	0.069	0.064	0.035	0.030	0.017	0.016	0.008	0.006
vi)	n=50	0.075	0.068	0.034	0.030	0.018	0.011	0.004	0.003
	n=100	0.094	0.092	0.051	0.048	0.029	0.025	0.010	0.009

Table 8

Estimated significance levels for approximate critical values for (a) R_m and (b) R^*_m statistics, based on MCS fit.

		α=0.10		α=0.05		α=0.025		α=0.01	
Configuration		(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
	n=25	0.091	0.076	0.045	0.044	0.020	0.023	0.007	0.009
i)	n=50	0.091	0.076	0.049	0.043	0.031	0.024	0.013	0.008
	n=100	0.090	0.075	0.050	0.042	0.028	0.022	0.014	0.008
	n=25	0.101	0.089	0.051	0.041	0.029	0.022	0.013	0.011
ii)	n=50	0.092	0.082	0.053	0.042	0.028	0.027	0.017	0.013
	n=100	0.082	0.076	0.043	0.037	0.026	0.021	0.015	0.007
	n=25	0.082	0.074	0,040	0,034	0,023	0.016	0.007	0.006
iii)	n=50	0.094	0.082	0.052	0.037	0.023	0.016	0.009	0.005
	n=100	0.095	0.074	0.051	0.033	0.024	0.017	0.01 1	0.009
	n-25	0.080	0.080	0.037	0.036	0.018	0.018	0.010	0.007
iv)	n=50	0.077	0.068	0.031	0,029	0.016	0,013	0.007	0.005
	n=100	0.084	0.081	0.044	0.043	0.022	0.021	0.009	0.008
	n=25	0.085	0.077	0.045	0.039	0.021	0.018	0.008	0.004
v)	n=50	0.092	0.093	0.045	0.042	0.018	0.016	0,006	0.007
	n=100	0.090	0.087	0.045	0.043	0.026	0.022	0.011	0.010
	n=25	0.064	0.058	0.028	0.031	0.015	0.013	0.005	0.004
vi)	n=50	0.082	0.075	0.036	0.035	0,015	0.010	0.003	0.003
	n=100	0.097	0.092	0.053	0.048	0.025	0.020	0.008	0.005

Table 9

Estimated significance levels for approximate critical values for (a) R_{max} and (b) R^*_{max} statistics, based on WLS fit.

Configuration		α=0.10 (a)	(b)	a=0.05 (a)	(b)	α=0.025 (a)	(b)	α=0.01 (a)	(b)
	n=25	0.052	0.039	0.023	0.017	0.013	0.008	0.004	0.004
i)	n=50	0.069	0.052	0.033	0.026	0.017	0.009	0.009	0.004
	n=100	0.072	0.058	0.038	0.026	0.020	0.012	0.008	0.006
	n=25	0.111	0. 104	0.061	0.057	0.035	0.032	0.018	0.015
ii)	n=50	0.095	0,088	0.053	0.048	0.028	0.024	0.012	0.011
	n=100	0.099	0.088	0.047	0.045	0.024	0.021	0.012	0.010
	n=25	0.199	0. 188	0. 121	0.112	0.073	0.080	0.039	0.047
iii)	n=50	0.157	0. 146	0.085	0.090	0.056	0.049	0.025	0.023
	n=100	0.130	0. 1 19	0.080	0.069	0.048	0.037	0.019	0.017
	n=25	0.076	0.076	0.037	0.038	0.019	0.020	0.009	0.006
iv)	n=50	0.070	0.064	0.030	0.028	0.016	0.013	0.006	0.005
	n=100	0.087	0.082	0.045	0.042	0.022	0.023	0.01 1	0.012
	n=25	0.108	0.104	0.057	0.057	0.034	0.030	0.014	0.013
v)	n=50	0.111	0.109	0.056	0.058	0.027	0.026	0.009	0.011
	n=100	0.104	0. 103	0.052	0.051	0.027	0.024	0.012	0.01 1
	n=25	0.1 16	0.116	0.059	0.053	0.023	0.024	0.010	0.01 1
vi)	n=50	0.1 12	0. 121	0.062	0.062	0.030	0.032	0.013	0.010
	n=100	0.1 12	0.111	0.055	0.050	0.028	0.027	0.01 1	0.008

Table 10

Estimated significance levels for approximate critical values for (a) R_{min} and (b) $R^*_{min.}$ statistics, based on WLS fit.

Configuration	α = 0.10 (a)	(b)	α=0.05 (a)	(b)	α=0.025 (a)	(b)	α=0.01 (a)	(b)
n=25	0.152	0. 157	0.009	0.097	0.058	0.063	0.031	0.036
i) n=50	0.137	0. 133	0.076	0.069	0.043	0.046	0.025	0.021
n=100	0.113	0.113	0.064	0.064	0.039	0.037	0.018	0.017
n=25	0.104	0.104	0.063	0.054	0.037	0.029	0.014	0.014
ii) n=50	0.094	0.090	0.053	0.052	0.032	0.027	0.017	0.014
n=100	0.096	0.089	0,048	0.045	0.025	0.022	0.013	0.010
n=25	0.039	0.028	0.018	0.013	0.011	0.006	0.005	0.002
iii) n=50	0.056	0.039	0.027	0.016	0.013	0.006	0.003	0.002
n=100	0.053	0.042	0.026	0.017	0.010	0.007	0.005	0.002
n=25	0.125	0.125	0.065	0.064	0.035	0.035	0.013	0.014
iv) n=50	0.117	0.112	0.061	0.053	0.023	0.022	0.009	0.009
n=100	0.102	0.104	0.048	0.04 7	0.023	0.022	0,006	0.006
n=25	0.075	0.076	0.039	0.040	0.020	0.023	0.011	0.009
v) n=50	0.093	0.098	0.050	0.049	0.024	0.022	0.008	0.004
n=100	0.097	0.093	0.047	0.044	0.025	0.025	0.013	0.011
n=25	0.061	0.056	0.031	0.030	0.015	0.015	0,008	0.006
vi) n=50	0.071	0.064	0.035	0.029	0.019	0.012	0.004	0.003
n=100	0.094	0.091	0.050	0.049	0.029	0.025	0.010	0.010

Table 11

Estimated significance levels for approximate critical values for (a) R_m and (b) R^*_m statistics, based on WLS fit.

	α = 0. 10		α=0.05		α=0.025		a=0.01	
Configuration	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
n=25	0.115	0. 104	0.068	0.067	0.040	0.044	0.023	0.021
i) n=50	0,102	0.083	0.058	0.052	0.038	0.029	0.018	0,013
n=100	0.092	0.078	0.055	0.044	0.031	0.022	0.016	0.010
n=25	0.113	0.010	0.066	0.058	0.035	0.032	0.016	0.019
ii) n=50	0.095	0.085	0.056	0.046	0.033	0.029	0.019	0.014
n=100	0.085	0.078	0.045	0.039	0.028	0.022	0.015	0.007
n=25	0.134	0. 124	0.081	0.083	0.052	0.054	0.023	0.032
iii) n=50	0.106	0.097	0.067	0.052	0.034	0.028	0.016	0.012
n=100	0.101	0.077	0.057	0.039	0.028	0.020	0.013	0.010
n=25	0.100	0.099	0.053	0.056	0.026	0.028	0.013	0.012
iv) n=50	0.089	0.076	0.038	0.034	0.019	0.016	0.008	0.008
n=100	0.090	0.086	0.045	0.044	0.024	0.023	0.010	0.009
n=25	0.092	0.093	0.052	0.051	0.030	0.028	0.01 1	0.009
v) n=50	0.099	0.099	0.050	0.046	0.022	0.019	0.007	0.007
n=100	0.093	0.088	0.049	0.046	0.028	0.025	0.012	0.011
n=25	0.088	0.079	0.037	0.036	0.017	0.020	0.007	0.006
vi) n=50	0.094	0.088	0.048	0.044	0.023	0.020	0.004	0.006
n=100	0.101	0.096	0.056	0.050	0.026	0.023	0.008	0.008

Table 12

Estimated significance levels for approximate critical values for (a) R_{max} and (b) R^*_{max} statistics, based on MWLS fit.

	α=0. 10		α=0.05		α=0.025		α=0.01	
Configuration	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
n=25	0.046	0.036	0.022	0.017	0.013	0.007	0.004	0.003
i) n=50	0.059	0.043	0.026	0.021	0.016	0.008	0.006	0.003
n=100	0.060	0.046	0.030	0.022	0.016	0.011	0.008	0.005
n=25	0.111	0.108	0.061	0.058	0.034	0.032	0.018	0.014
ii) n = 50	0.099	0.090	0.053	0.050	0.030	0.024	0.012	0.01 1
n=100	0.095	0.090	0.049	0.04 7	0.025	0.022	0.012	0.01 1
n=25	0.183	0.166	0.089	0.085	0.044	0.040	0.019	0.016
iii) n=50	0.177	0.131	0.090	0.088	0.054	0.046	0.024	0.018
n=100	0.151	0.136	0.089	0.082	0.054	0.043	0.022	0.019
n=25	0.075	0.066	0.033	0.036	0.017	0.018	0.010	0.090
iv) n=50	0.053	0.052	0.025	0.025	0.012	0.011	0.005	0.005
n=100	0.070	0.069	0,035	0.036	0.020	0.021	0.010	0.010
n=25	0.110	0.108	0.058	0.056	0.032	0.028	0.012	0.010
v) n=50	0.117	0.116	0.062	0.060	0.031	0.028	0.009	0.01 1
n=100	0.112	0. 109	0.056	0.055	0.030	0.025	0.013	0.01 1
n=25	0.110	0.098	0.045	0.040	0.020	0.020	0.009	0.008
vi) n = 50	0.134	0.134	0.070	0.067	0.030	0.032	0.01 1	0.010
n=100	0.134	0.131	0.064	0.057	0.035	0.030	0.01 1	0.011

Table 13

Estimated significance levels for approximate critical values for (a) R_{min} and (b) R^*_{min} statistics, based on MWLS fit.

	α=0.10		α=0.05		α=0.025		a=0.01	
Configuration	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
n=25	0.150	0.15 1	0.092	0.081	0.049	0.047	0.019	0.022
i) n=50	0. 155	0. 149	0.088	0.073	0.045	0.046	0.026	0.022
n=100	0.135	0.133	0.075	0.071	0.044	0.042	0.020	0.018
n=25	0.105	0. 109	0.059	0.053	0.035	0.026	0.012	0.013
ii) n=50	0.102	0.089	0.055	0.053	0.032	0.030	0.017	0.014
n=100	0.095	0.090	0.050	0.045	0.025	0.024	0.014	0.010
n=25	0.038	0.033	0.021	0.016	0.012	0.008	0.007	0.003
iii) n=50	0.04 7	0.037	0.022	0.015	0.010	0.004	0.003	0.002
n=100	0.042	0.030	0.018	0.013	0.009	0.007	0.004	0.002
n=25	0.124	0. 123	0.061	0.058	0.030	0.026	0.011	0.012
iv) n=50	0.133	0. 128	0.067	0.061	0.028	0.023	0.010	0.009
n=100	0.1 19	0.1 14	0.057	0.055	0.027	0.024	0.009	0.008
n=25	0.069	0.070	0.036	0.039	0.020	0.022	0.009	0.007
v) n=50	0.090	0.091	0.047	0.046	0.023	0.019	0.007	0.003
n=100	0.087	0.085	0.044	0.043	0.023	0.024	0.012	0.010
n=25	0.055	0.054	0.031	0.027	0.015	0.015	0.006	0.008
vi) n=50	0.057	0.051	0.025	0.025	0.013	0.010	0.003	0.002
n=100	0.080	0.074	0.043	0.040	0.023	0.019	0.008	0.006

Table 14

Estimated significance levels for approximate critical values for (a) R_m and (b) R^*_m statistics, based on MWLS fit.

	α=0.10		α=0.05		α=0.025		α=0.01	
Configuration	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
n=25	0.107	0.087	0.058	0.051	0.029	0.030	0.011	0.014
i) n=50	0.107	0.084	0.059	0.051	0.039	0.028	0.016	0.012
n=100	0.095	0.081	0.057	0.048	0.033	0.025	0.016	0.011
n=25	0.109	0.099	0.065	0.053	0.036	0.030	0.016	0.015
ii) n=50	0.097	0.088	0.058	0.050	0.032	0.030	0.019	0.014
n=100	0.088	0.080	0.046	0.043	0.029	0.022	0.015	0.009
n=25	0.103	0.090	0.054	0.044	0.028	0.024	0.012	0.009
iii) n=50	0.105	0.093	0.062	0.049	0.034	0.023	0.012	0.007
n=100	0.102	0.087	0.062	0.045	0.034	0.022	0.015	0.011
n=25	0.092	0.091	0.046	0.043	0.023	0.022	0.011	0.090
iv) n=50	0.090	0.082	0.039	0.033	0.018	0.016	0.008	0.007
n=100	0.091	0.088	0.047	0.044	0.023	0.023	0.010	0.008
n=25	0.090	0.091	0.050	0.049	0.026	0.025	0.009	0.008
v) n=50	0.102	0.101	0.051	0.045	0.022	0.017	0.007	0.007
n=100	0.093	0.091	0.050	0.046	0.028	0.025	0.012	0.011
n=25	0.073	0.065	0.034	0.033	0.017	0.018	0.007	0.006
vi) n=50	0.092	0.088	0.042	0.042	0.019	0.017	0.003	0.005
n=100	0.103	0.094	0.057	0.047	0.024	0.023	0.008	0.008

References

Al-Sarraf, Z, and Young, D.H. (1985). Properties of estimators of parameters in logistic regression models. Technical Report TR/05/85, Department of Mathematics and Statistics, Brunel University.

Berkson, J. (1955). Maximum likelihood and minimum x^2 estimates of the logistic function. J. Amer. Statist. Assoc., 50, 130-162.

Cox, D.R. and Snell, E.J. (1968). A general definition of residuals. J. R. Statist. Soc. B, 30, 248-265.

Gart, J.J. and Zweifel, J.R. (1967). On the bias of various estimators of the logistic function and its variance with application to quantal bioassay. Biometrika, 54, 181-187.